Correlation structure of the δ_n statistic for chaotic quantum systems

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The existence of a formal analogy between quantum energy spectra and discrete time series has been recently pointed out. When the energy level fluctuations are described by means of the δ_n statistic, it is found that chaotic quantum systems are characterized by $1/f$ noise, while regular systems are characterized by $1/f^2$. In order to investigate the correlation structure of the δ_n statistic, we study the qth-*order height-height correlation function* $C_q(\tau)$, which measures the momentum of order q, i.e., the average qth power of the signal change after a time delay τ. It is shown that this function has a logarithmic behavior for the spectra of chaotic quantum systems, modeled by means of random matrix theory. On the other hand, since the power spectrum of chaotic energy spectra considered as time series exhibit $1/f$ noise, we investigate whether the qth-order height-height correlation function of other time series with $1/f$ noise exhibits the same properties. A time series of this kind can be generated as a linear combination of cosine functions with arbitrary phases. We find that the logarithmic behavior arises with great accuracy for time series generated with random phases.

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I. INTRODUCTION

The statistical study of energy level fluctuations is one of the most important tools for understanding quantum chaos. The pioneering work of Berry and Tabor [1], and Bohigas *et al.* [2] showed that spectral fluctuations have universal statistical properties, which are different for quantum systems with integrable or chaotic classical analogues [3]. In the first case, the sequence of spacings between consecutive energy levels constitutes a noncorrelated random sequence, whereas systems with chaotic classical analogues are characterized by strong level correlations well described by random matrix theory (RMT). Therefore, a generic quantum system (with or without a clear classical analogue) is usually said to be integrable or chaotic when its statistical spectral properties coincide with those of a noncorrelated sequence or those of random matrix theory, respectively. A comprehensive review of these features and later developments in this line can be found in Refs. [4,5].

Recently, a different approach to the study of spectral fluctuations has been proposed [6]. It was noted that there is a formal similarity between the discrete spectrum of quantum systems and a discrete time series. Considering the sequence of energy levels as a signal in which energy plays the role of time, level fluctuations can be studied using traditional methods of time series analysis, especially regarding the behavior of the power spectrum in Fourier space. Using an appropriate statistic called δ_n, defined as the accumulated departure of the spacing sequence from its mean value, it was shown by numerical calculations in atomic nuclei and paradigmatic random matrix ensembles that chaotic quantum systems are characterized by $1/f$ noise, whereas integrable quantum systems exhibit $1/f^2$ noise [6]. As a consequence, this behavior was conjectured to be a general property of all chaotic or regular quantum systems. Some understanding of this peculiar behavior can be achieved considering the existing analogy between the characteristic level repulsion of chaotic quantum systems and the antipersistent features of a time series characterized by $1/f$ noise, on the one hand [7], and between the uncorrelated level fluctuations of integrable quantum systems and uncorrelated displacements in Brownian motion, on the other hand.

Following this line of work, analytical expressions for the power spectrum of the δ_n statistic were derived from random matrix theory and semiclassical theory [11], and the previously conjectured $1/f$ and $1/f^2$ types of noise were thus theoretically confirmed for chaotic and regular quantum systems, respectively.

A natural question to ask is what happens in intermediate situations between chaotic and regular motion. The transition between these two extremes was studied using the Robnik quantum billiard, in which it takes place very smoothly as the billiard shape changes. Amazingly, it was found that the power spectrum of δ_n exhibits a fractional power law behavior, usually called $1/f^{\alpha}$ noise, through the whole transition, with α smoothly changing from the extreme value $\alpha=1$ in the chaotic regime to $\alpha=2$ in the regular one [12]. A similar result was recently obtained for a coupled quartic oscillator and a quantum top [13].

It is our purpose in this paper to get a deeper understanding of the connection between the fluctuation properties of quantum levels and time series. To this end, we go beyond the comparison of their power spectra. By calculating the qth-order height-height correlation function for higher order moments, we study whether there is a multiscaling structure in classical random matrix ensembles. We do not find such scaling behavior, but a logarithmic correlation structure, and we propose an heuristic expression which provides an accurate description of the qth-order height-height correlation function for all the spectral moments. We also investigate the same function for some time series with $1/f$ noise, and we compare it to the results for random matrix ensembles.

In Sec. II, we introduce the concepts of qth-order height-height correlation function, self-affinity of a time series, and emphasize the properties of the second moment for the particular case of $1/f$ noise. In Sec. III, we introduce the δ_n

statistic and outline the properties of its power spectrum. In Sec. IV, we present the calculations, explain how the qth-order height-height correlation function is calculated, and discuss the results. Section V is devoted to the comparison with time series which exhibit $1/f$ noise and are generated as linear combination of cosine functions. Finally, Sec. VI contains a summary and the conclusions.

II. SELF-AFFINE TIME SERIES AND 1/F NOISE

Given a continuous time series $X(t), t \in \mathbb{R}$, the power spectrum of the signal is defined as

$$P(\omega) = |\hat{X}(\omega)|^2, \tag{1}$$

where

$$\hat{X}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(t) \exp(-2\pi i \omega t) \tag{2}$$

is the Fourier transform of the signal. It is well known that many phenomena in nature and social sciences can be characterized by time series with power spectra of the form

$$P(\omega) = C\omega^{-\alpha}, \tag{3}$$

where C is a positive constant and $\alpha \geqslant 1$ (see, for example, Refs. [14,10] and references therein). This frequency decomposition entails that such a time series, generically known as $1/f^{\alpha}$ noise, has no characteristic time scale, and its correlation time is comparable to the duration of the entire time series. Moreover, it is quite customary to assume self-affinity, i.e., that in addition to the previously mentioned characteristics, a $1/f^{\alpha}$ noise has a multiscaling structure in all time scales [15].

For a continuous time series $X(t)$, the so-called qth-order height-height correlation function [16] (which we denote qth-order correlation function, for the sake of simplicity), is given by

$$C_q(\tau) = \langle |X(t) - X(t+\tau)|^q \rangle_t = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt |X(t+\tau) - X(t)|^q. \tag{4}$$

In terms of the qth-order correlation function, multiscaling means that

$$C_q(\tau) \propto \tau^{qH_q}, \tag{5}$$

where H_q is a smooth function of q. The shape of H_q determines the character of the scaling. When $H_q = H_0$ for all q, the scaling is said to be simple, because it is the same for all momenta; fractional Brownian motion (FBM) is the paradigmatic example of this behavior [8]. On the other hand, if H_q is not a constant function, one talks of nontrivial multiscaling; a good example of this scaling behavior is proposed by Mandelbrot [17]. It is noteworthy to comment that the statistical properties of a random signal are fully determined by all $q = 1, 2, \ldots, \infty$ momenta [18], and thus the study of $C_q(\tau)$ provides us a lot of information about the signal.

The power spectrum (1) is related to the second-order correlation function as [19]

$$C_2(\tau) = \frac{2}{\pi} \int_0^{\infty} d\omega\, P(\omega)[1 - \cos(\omega\tau)]. \tag{6}$$

When the power spectrum is given by Eq. (3), this integral is well defined for $1 < \alpha < 3$, and $C_2(\tau) \propto \tau^{\alpha-1}$, as expected for a self-affine time series.

Nevertheless, in spite of the usual scaling assumptions, the power law (3) is not a necessary condition to have self-affine time series. Actually, studying some statistical properties related to self-affinity, like the fractal dimension of the signal, Greis and Greenside [10] have found systematic deviations from self-affinity in time series that exhibit $1/f^{\alpha}$ noise when $\alpha \neq 2$.

Moreover, for $\alpha = 1$, there also exist analytical arguments showing that these time series are not self-affine. Since $1/f$ noise characterizes the spectral fluctuations of chaotic quantum systems, this case is very important. The relation between $C_2(\tau)$ and $P(\omega)$ does not hold if $\alpha = 1$. In fact, the integral appearing in the right-hand side (rhs) of Eq. (6) diverges. Moreover, it is not possible to avoid this problem by redefining the correlation function; the divergence in the second-order momentum is characteristic of $1/f$ noise [20,21].

To overcome this difficulty, one usually introduces a discrete time series by sampling the original signal with a characteristic time Δt. (Note that time series are usually sampled, because numerical calculations and experimental measurements involve some sort of discretization.) The sampling defines an upper frequency limit in the Fourier spectrum $\omega_{max} = 2\pi/\Delta t$, and thus, we can write

$$C_2(\tau) = \frac{2}{\pi} \int_0^{\omega_{max}} d\omega\, P(\omega)[1 - \cos(\omega\tau)], \tag{7}$$

where the integral in the rhs is well defined for $P(\omega) \propto \omega^{-1}$ and leads to

$$C_2(\tau) = \frac{2}{\pi} \left[\gamma - \mathrm{Ci}\left(\frac{2\pi\tau}{\Delta t}\right) + \ln \tau + \ln\left(\frac{2\pi}{\Delta t}\right) \right], \tag{8}$$

where $\gamma = 0.577\,216$ is the Euler constant, and $\mathrm{Ci}(x)$ is the cosine integral function [22]. A good approximation for this expression is

$$C_2(\tau) \sim \ln \tau, \quad \frac{\tau}{\Delta t} \gg 1. \tag{9}$$

This result shows that a strict $1/f$ noise is not self-affine, since at least the second-order correlation function does not have a multiscaling behavior; the leading term is a logarithm and not a power law. In Ref. [23], a similar result was obtained by studying several mechanisms that give rise to times series that exhibit $1/f$ noise.

Beyond the sampling of continuous time series involved in numerical calculations or experiments, discrete time series are important on their own. In our case, the discrete energy spectrum of a quantum system is related to a discrete time series; for this reason, it is worth it to translate the previous expressions for discrete time series. Given a discrete and finite time series $X(n)$, $n=1,2,\ldots,N$, the power spectrum, defined only in a finite set of frequencies $\omega_k=2\pi k/N$ with $k=1,2,\ldots,N/2$, is given by

$$P(\omega_k) = |\hat{X}(\omega_k)|^2, \tag{10}$$

in terms of the discrete Fourier transform of the signal

$$\hat{X}(\omega_k) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X(n)\exp(-i\omega_k n). \tag{11}$$

For such a time series, the qth-order correlation function is

$$C_q(n) = \langle |X(m) - X(m+n)|^q \rangle_m = \frac{1}{N'} \sum_{m=1}^{N'} |X(m) - X(m+n)|^q, \tag{12}$$

where N' is the number of points over which the moving average is taken and satisfies that $1 \ll N' \leqslant N-n$.

III. QUANTUM CHAOS AND TIME SERIES

It has been pointed out that energy level spectra have a formal similarity with time series; this analogy is the basis of all the results presented in Refs. [6,11,12]. In order to understand this point, let us consider a physical system whose Hamiltonian is self-adjoint, time independent, bounded from below, and with discrete spectrum $\{E_n, n \in \mathbb{N}\}$. It is well known that its energy level density $g(E)=\Sigma_n \delta(E-E_n)$ can be written as

$$g(E) = \bar{g}(E) + \tilde{g}(E), \tag{13}$$

where $\bar{g}(E)$ is a smooth function of the energy and $\tilde{g}(E)$ represents the rapidly fluctuating part [24]. For chaotic and regular systems, the latter is universal to a large extent; it only reflects the character of the underlying classical dynamics, regardless of the system specificities [25]. Now, defining

$$\delta_n = \int_{-\infty}^{E_{n+1}} \tilde{g}(\epsilon) d\epsilon, \tag{14}$$

considering the order index n as a discrete time, and δ_n as the value of the fluctuation at time $t=n$, the sequence $\{\delta_n\}$ can be treated as a time series. In general, it is convenient to define δ_n to be independent of the ground state of the system, because in many experiments and numerical calculations only a window of energy levels is available; therefore, if E_1 is the first known energy level, the δ_n sequence can be defined as follows

$$\delta_n = \int_{E_1}^{E_{n+1}} \tilde{g}(\epsilon) d\epsilon. \tag{15}$$

The δ_n function can be alternatively defined as the fluctuation of the excitation energy of the unfolded levels [6].

Certainly, there are some differences between δ_n and actual time series (that is, physical magnitudes evolving in time); the most important is that in a quantum system the position of each energy level depends both on lower and higher energy levels, and thus, δ_n as a time series depends on the past as well on the future history. Nevertheless, in spite of this and other peculiarities (see Ref. [6] for a complete discussion), the analogy between an energy level spectrum and a time series is well established.

The analysis of the power spectrum of δ_n, considered as a time series, has given rise to the following result: energy level fluctuations of chaotic quantum systems are characterized by $1/f$ noise, whereas those of quantum integrable systems show $1/f^2$ noise [6,11], i.e.,

$$P^{\delta}(\omega_k) \propto \begin{cases} \dfrac{1}{\omega_k} & \text{for chaotic systems,} \\[2ex] \dfrac{1}{\omega_k^2} & \text{for integrable systems.} \end{cases} \tag{16}$$

Moreover, for some intermediate (neither integrable, nor chaotic) Hamiltonians, a $1/f^{\alpha}$ noise with $1<\alpha<2$ has been found [12,13]

$$P^{\delta}(\omega_k) \propto \frac{1}{\omega_k^{\alpha}}. \tag{17}$$

IV. CALCULATIONS AND RESULTS

In this paper, we use RMT to model quantum chaos. It plays a predominant role in the description of chaotic quantum systems; in spite of a few counterexamples, it is the accepted theory for describing the energy level fluctuations of chaotic quantum systems in the universal regime [4,5]. Although there does not exist a general and widely accepted proof of this empiric result, important efforts are going to show that RMT and the semiclassical approximation give rise to the same results in a wide range of scales [27].

Here, we deal with two classical random matrix ensembles (CRMEs): the Gaussian orthogonal ensemble (GOE) and the Gaussian unitary ensemble (GUE) of N-dimensional matrices. The former is applicable for time-reversal invariant chaotic systems with rotational symmetry or with broken rotational symmetry and integer spin; and the latter is applicable for quantum chaotic systems without time-reversal invariance. They describe the universal regime of the spectral fluctuations for all known chaotic quantum systems [28].

In order to calculate the qth-order correlation function defined in Eq. (12), we perform a twofold average: first a moving average over the single spectrum $\langle \bullet \rangle$, and afterwards an ensemble average $\overline{\bullet}$ over different members of the same ensemble

$$C_q(n) = \overline{\langle \delta_{m+n} - \delta_m \rangle_m}. \tag{18}$$

This double average greatly reduces the statistical fluctuations of $C_q(n)$ (specially for high values of q), and thus allows us to obtain significant results for a wide range of q values. In practice, the ensemble average is performed with a finite number M of realizations of N-dimensional spectra. The moving average is carried out by summing over the first

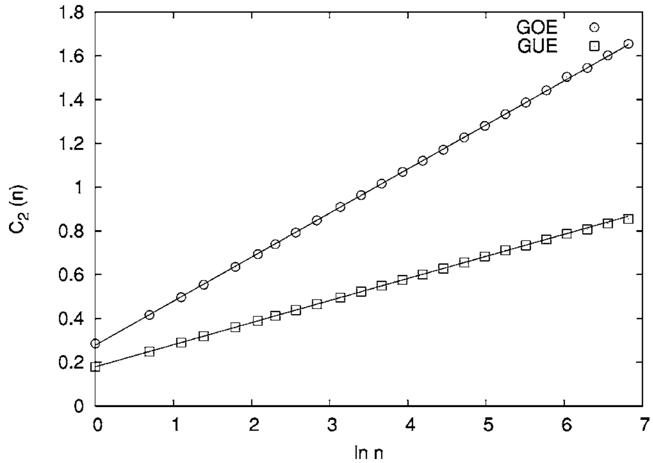

FIG. 1. Numerical values of $C_2(n)$ for GOE (open circles) and GUE (open squares), compared to the best fit of Eq. (21) to the calculated values (solid lines)

$3N/4$ levels of the spectrum. This makes it possible to include many different regions of the spectrum and obtain reliable statistical results. Therefore, Eq. (18) can be written as

$$C_q(n) = \frac{1}{M}\frac{4}{3N}\sum_{M}\sum_{m=1}^{3N/4} |\delta_{m+n} - \delta_m|^q, \quad n = 1,2,\ldots,N/4. \tag{19}$$

In this paper, the (double) numerical averages have been calculated by using $M=500$ ensemble matrices of dimension $N=4096$.

A. The second-order momentum

According to Eqs. (7) and (16), $C_2(\tau)$ should behave logarithmically for quantum chaotic systems. Using analytical methods, Bohigas, Leboeuf, and Sánchez have calculated this function for the three CRME [26], and their main results are that

TABLE I. Best fit values of the coefficients A_2 and B_2 (numeric) compared to the predictions of Bohigas *et al.* (theory).

	A_2		B_2	
	Theory	Numeric	Theory	Numeric
GOE	0.2026	0.2013 (3)	0.2753	0.278 (1)
GUE	0.1013	0.1009 (1)	0.1794	0.1794 (1)

$$C_2(n) = \frac{2 \ln n}{\beta\pi^2} + \frac{2[\gamma + \ln(2\pi) + 1]}{\beta\pi^2} + \frac{\beta}{4} - \frac{2}{3} + O(1/n^2),$$

$$n \in \mathbb{N}, \tag{20}$$

where $\gamma=0.577\,216$ is the Euler constant, and $\beta=1$, 2 for GOE and GUE, respectively.

As a preliminary test of our calculation, we try to reproduce their result. Figure 1 shows $C_2(n)$ for GOE and GUE using a semilogarithmic scale. Indeed, the calculated points fit very precisely to the law

$$C_2(n) = A_2 \ln n + B_2, \tag{21}$$

almost through the whole range of n values, i.e., for $n \leqslant N/4$. As it can be seen in Table I, the values of A_2 and B_2, obtained by means of a least-squares fit, agree with the theoretical predictions of Eq. (20) with a precision higher than 1%.

B. *q*th-order momentum

Figure 2 shows the behavior of $C_q(n)$ in a wide interval of q values. For the sake of clarity, the figure is divided in four subpanels with different types of scales; this makes it possible to distinguish properly the main trend of the correlation function through the whole q interval studied in this work. Upper panels display the values of the qth-order correlation function for GOE while lower ones do for GUE. Using different symbols and a double logarithmic scale, left panels

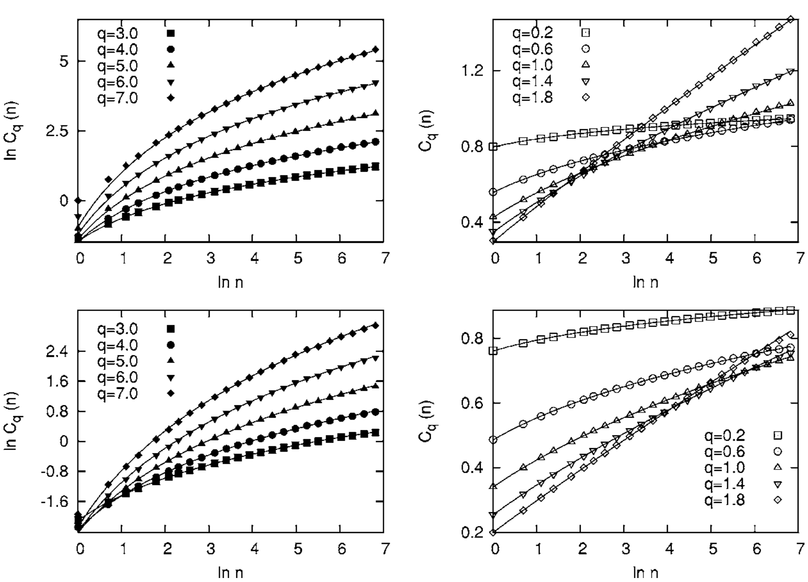

FIG. 2. Numerical behavior of $C_q(n)$ for GOE (upper panels) and GUE (lower panels) calculated using 500 matrices of dimension 4096 for each ensemble. Left panels display the results for q=3, 4, 5, 6, and 7 using a log-log scale. Right panels show $C_q(n)$ for q=0.2, 0.6, 1.0, 1.4, and 1.8 in a logarithmic scale.

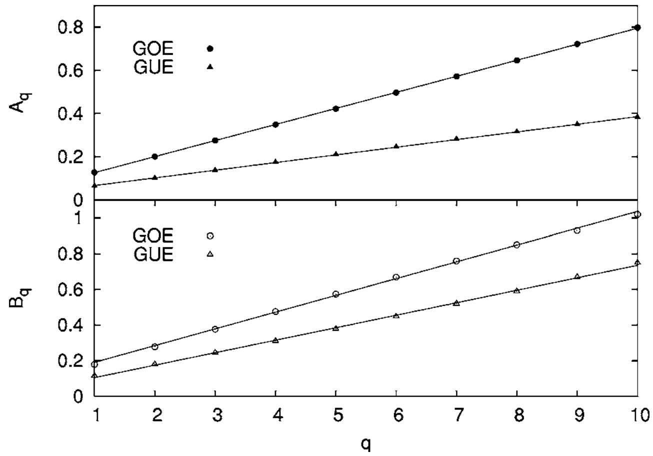

FIG. 3. Behavior of the coefficients A_q and B_q [defined in Eq. (22)] for GOE (circles) and GUE (triangles). Their values are obtained by a least-squares fit of Eq. (22) to the calculated values of $C_q(n)$. Solid lines represent a least-squares fit of A_q and B_q values to a linear law in q.

show the behavior of $C_q(n)$ for $3 \leqslant q \leqslant 7$, while the right panels display $C_q(n)$ vs $\ln n$ when $0.2 \leqslant q \leqslant 1.8$. Solid lines represent the result of a linear-square fit of the calculated values to the law

$$C_q(n) = (A_q \ln n + B_q)^{q/2}, \tag{22}$$

which is a generalization of Eq. (21). The agreement between the numerical values of $C_q(n)$ and the continuous curves is excellent except for very small values of n. In this region, the discrepancies between the actual $C_q(n)$ values and the predictions of Eq. (22) seem to be larger for GOE spectra. This departure from a logarithmic behavior was already known for the $q=2$ momentum. Actually, the result of Bohigas *et al.* (20) includes corrections of order $O(1/n^2)$ that become significant when n is small. These additional terms of $C_q(n)$ are related to the fact that in RMT, the power spectrum of δ_n is not an exact power law; there exist small but clear deviations at high frequencies. Moreover, the deviations of $P(\omega)$ from an exact $1/f$ noise are larger for GOE than for GUE spectra, and therefore, we can expect Eq. (22) to be a better approximation for the latter. Strictly speaking, these arguments are valid for $C_2(n)$ since we only have analytical results for this function and its relation to $P(\omega)$. Nevertheless, the results of Bohigas, Haq, and Pandey for the three-point and the four-point correlation functions in RME [29] suggest a similar behavior when $q > 2$.

In order to fully characterize the correlation structure of δ_n for chaotic quantum systems, we must set the shape of A_q and B_q inside of a q interval as large as possible. In this work, we have been able to obtain their values for $1 \leqslant q \leqslant 10$. Beyond this interval, we can still use the same method to obtain significant results, but larger dimensionalities and number of members of the matrix ensemble are needed. The values of A_q (upper panel) and B_q (lower panel) for GOE and GUE are displayed in Fig. 3. It is clearly seen that both functions increase linearly with q, and again the agreement between the calculated points and the fit to a linear function is excellent. This allows us to conclude that A_q and B_q are exact linear functions.

Collecting together these results, we can state the following.

Conjecture. Let H be a Hamiltonian matrix pertaining to the Gaussian orthogonal ensemble or to the Gaussian unitary ensemble. The correlation structure of its spectrum, considered as a time series, is characterized by a qth-order correlation function

$$C_q(n) = (A_q \ln n + B_q)^{q/2}, \quad \forall \ q \geqslant 0,$$

where A_q and B_q are linear functions of the exponent q.

An important consequence of this conjecture is that the scaling behavior of GOE and GUE spectra differs from a power law $C_q(n) \propto n^{qH_q}$. Therefore, the spectra of these ensembles, considered as time series, are not self-affine. Nevertheless, this result could be considered as a particular case of a more generic law

$$C_q(n) = (A_q \ln n + B_q)^{qH_q}, \tag{23}$$

where H_q is a smooth function of q. In our case $H_q = 1/2 \ \forall \ q$, a result that can be understood as a simple behavior, since all momenta can be characterized by means of a single exponent $H_q = 1/2 \ \forall \ q$—something similar happens for fractional Brownian motion, which is considered more simple than multifractal processes [30].

V. COMPARISON WITH OTHER MODELS

As is well known, $1/f^\alpha$ noise is ubiquitous. Many physical systems and mathematical models with this property are self-affine, i.e., their correlation function follows a power law. However, our previous result raises the question whether other systems (apart from CRME) have a logarithmic behavior. In order to obtain a partial answer to this question, we calculate the correlation structure of time series

$$X(n) = \sum_{k=1}^{N/2} \sqrt{\frac{1}{k}} \cos\left(\frac{2\pi k n}{N} + \phi_k\right), \tag{24}$$

with Eq. (23). This algorithm, proposed by Greis [10], generates an exact $1/f$ noise; the unique source of randomness is the set of phases ϕ_k. Different phase sequences might give rise to different correlation structure. For this reason, we have calculated $C_q(n)$ for an uniformly distributed, noncorrelated sequence of phases in the interval $[0, 2\pi)$, and a quasiperiodic sequence of phases

$$\phi_k = s_0 + \pi[\sin(k) + \sin(\sqrt{k})](\text{mod } 2\pi), \tag{25}$$

which densely fills out the interval $[0, 2\pi)$. In this case, different realizations are obtained by randomly selecting s_0 in the same interval [10].

Figure 4 compares $C_q(n)$ for both types of time series. In order to simplify the comparison, we only present the results for $q=2$ and $q=4$. Although $C_2(n)$ increases logarithmically regardless of the phase sequence (as it correspond to exact $1/f$ noises), $C_4(n)$ is an exact logarithmic function only for a pure random phase sequence.

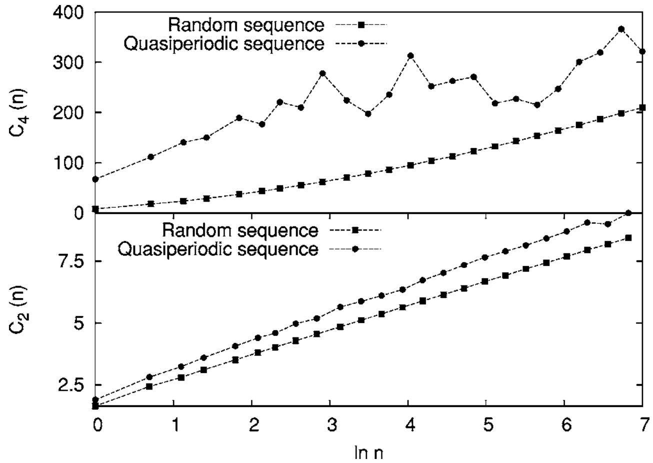

FIG. 4. $C_2(n)$ values (lower panel) and $C_4(n)$ values (upper panel) for time series generated with Eq. (24) and two different set of phases: random noncorrelated sequences, uniformly distributed in $[0,2\pi)$ (squares), and quasiperiodic sequences (25) (circles).

Therefore, we restrict our analysis to time series generated with Eq. (24) and random phase sequences. The qth-order correlation function $C_q(n)$ is shown in Fig. 5; in order to clarify its behavior for different values of q, we have

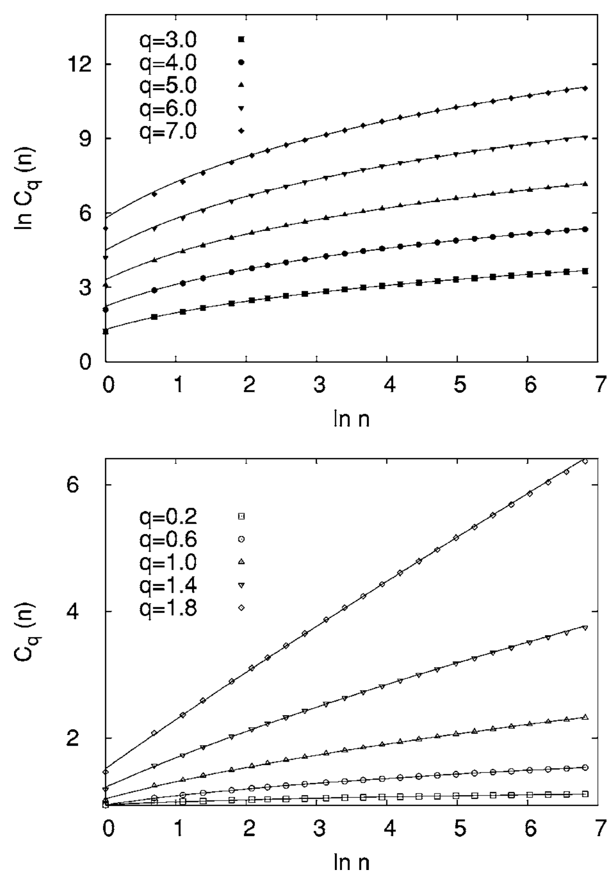

FIG. 5. Numerical behavior of $C_q(n)$ for time series generated with Eq. (24), calculated for selected values of q using 500 realizations of length 4096. The upper panel displays the results for $q=3$, 4, 5, 6, and 7 using a log-log scale. The lower panel shows $C_q(n)$ for $q=0.2$, 0.6, 1.0, 1.4, and 1.8 in a logarithmic scale.

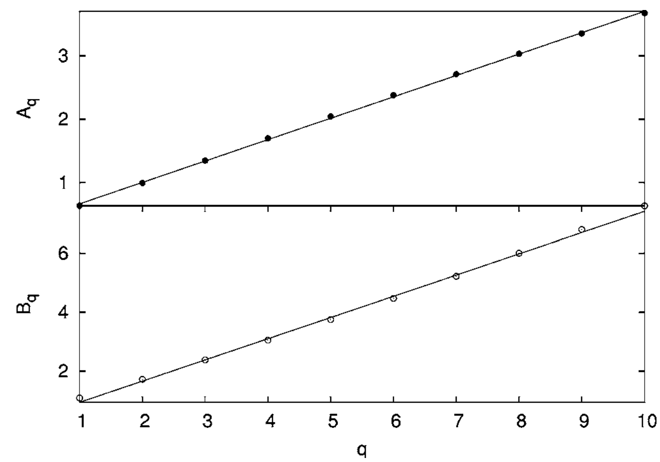

FIG. 6. Linear behavior of the coefficients A_q and B_q [defined in Eq. (22)] for time series generated with Eq. (24). Their values are obtained by a least-squares fit of Eq. (22) to the calculated values of $C_q(n)$. Solid lines represent a least-squares fit of A_q and B_q values to a linear law in q.

divided this figure in two panels. Using a double logarithmic scale, the upper panel displays the calculated values of the qth-order correlation function for $3 \leq q \leq 7$, while the lower one shows $C_q(n)$ versus $\ln n$ when $0.2 \leq q \leq 1.8$. Again, the continuous curves are the result of a least-squares fit to Eq. (22). Beyond any doubt, these results are very similar to those obtained for CRME. However, since we are now dealing with an exact $1/f$ noise, the agreement between the calculated points and the theoretical law is more accurate, even at very low n values.

In order to complete the comparison with RMT, we have calculated the coefficients A_q and B_q. Figure 6 proves that they are linear functions of the order index q. Although the slope is different for the energy level spectra of chaotic quantum systems and for these time series (generated as an appropriate linear combination of cosine functions), we can clearly state that their statistical properties do coincide.

VI. SUMMARY AND CONCLUSSIONS

Using the formal analogy between energy level spectra and time series, we have studied the correlation structure in the spectra of some classical random matrix ensembles, which can be considered as the paradigms of chaotic quantum systems. The qth-order correlation function $C_q(n)$ has been calculated for several q values in the range $0 < q \leq 10$. The calculations are performed using a twofold average, first over many sets of levels within the spectrum of a large dimensional matrix, and afterwards over a large number of different members of the matrix ensemble. This procedure ensures good accuracy for the qth-order correlation function calculations.

Inspired by the linear behavior in $\ln n$ of the second-order correlation function $C_2(n)$, we have tried to see if the qth-order correlation function exhibits a logarithmic behavior of the type $C_q(n) = (A_q \ln n + B_q)^{q/2}$, where A_q and B_q are linear functions of q. For all the numerical examples considered, we have found that the qth-order correlation function of the δ_n statistic for GOE and GUE random matrices exhibits this kind of behavior very accurately. The agreement of this function with the numerical data is excellent in all cases, for small, large and intermediate q values in the whole $0 < q \leq 10$ range considered. Therefore, it seems reasonable to conjecture the same behavior for all momenta with $q > 0$. We call this behavior *simple logarithmic behavior*, since $C_q(n)$ has the same functional structure as the second-order correlation function, regardless of the value of q.

An interesting question is whether the simple logarithmic behavior of the qth-order correlation function is a peculiarity of the classical random matrix ensembles or may appear in other systems as well. As a first step in the study of this question, we have again made use of the analogy between level spectra and time series. We know that fluctuations in the spectra of chaotic quantum systems and random matrix ensembles are characterized by $1/f$ noise. We also know that $1/f$ noise is very ubiquitous in nature: many physical systems and mathematical models have this kind of fluctuations. But do they have simple logarithmic behavior?

We have obtained some answers to this question studying the correlation structure of a set of time series generated as a linear combination of cosine functions, as shown in Eq. (24). These time series have an exact $1/f$ noise behavior independent of the choice of phases ϕ_k. We have calculated $C_q(n)$ using two different ways to generate ϕ_k, and we found a simple logarithmic scaling behavior for a noncorrelated sequence of phases uniformly distributed in the interval $[0,2\pi)$. Furthermore, in this case, the agreement of the calculated qth-order correlation function with the ansatz $C_q(n) = (A_q \ln n + B_q)^{q/2}$ is perfect.

Finally, we would like to point out that these results clearly suggest a neat way to check whether random matrix theory describes properly all the statistical properties of chaotic quantum systems. With some exceptions, like the comparison of the RMT predictions for the three-point and four-point correlation functions with the nuclear data ensemble, this test has been systematically carried out by many authors using statistics associated to the second-order momentum. However, higher-order correlation functions ($C_q(n)$, $q > 2$) are very sensitive to the details of the joint probability density characteristic of the system or ensemble. Therefore, they may provide a very stringent test of the famous Bohigas-Giannoni-Schmit conjecture about the fluctuation properties of chaotic quantum systems.

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